Article – Mean Field Game Theory
Mean Field Game with Major and Minor players and applications to LQ games and Flocking models

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Introduction – Motivation

- Recent interest in models displaying interaction between agent's states and their distributions
  1. Mean Field Games (MFG)
  2. Control of McKean-Vlasov system (MKV)
  3. Mean Field Games with Major and Minor Players (MFG Maj-Min)
     $\Rightarrow$ Mixture of the two first systems
- In these contexts, the control problem is non-standard : need to develop new methods and theoretical results.

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     $\Rightarrow$ Mixture of the two first systems
- In these contexts, the control problem is non-standard : need to develop new methods and theoretical results.
- This article :
  - "*An alternative approach to Mean Field Game with Major and Minor players*"
  - R. Carmona and P. Wang
  - Develop probabilistic methods (FBSDE) applied to MFG Maj-Min
  - from the Stochastic Pontryagin Maximum Principle.
  - Tools borrowed from R. Carmona and X. Zhu (2016)

Introduction – What difference with usual MFGs ?

- Games with $N+1$ players, but when $N \to \infty$, the influence of the major player does not fade away in the asymptotic regime
- Asymmetry between the major player :
  - That should consider *its influence* on other players
  - Control of McKean Vlasov dynamics
- ... and the minor players
  - That consider a standard control problem, but *conditional* to the major player moves
  - Analogy with MFG with common noise.
- Link between the two problems in the search for a Nash Equilibrium (i.e. a fixed point !).

Introduction – Literature

- General case treated in R. Carmona and X. Zhu (2016)
  - At a cost of some formalism for the control of McKean Vlasov :
    - Major player : consider the derivative w.r.t. measure
- This type of problem introduced by Huang (infinite horizon) and Huang and Nguyen (finite horizon).
  - Non-linear case in Caines and Nourian.
  - Bensoussan, Yam, Chau
  - All these articles use PDE approach.
- Problem : in these articles, the major player states (and controls) do not enter the minor players' dynamics
  - Relaxing this assumption poses technical difficulties :
  - Huang and Nguyen : anticipative variational calculus
  - R. Carmona and X. Zhu (2016) : Pontryagin Principle
- This article – R. Carmona and P. Wang (2017) :
  - Treats the (simple) case, LQ, and provide an application
  - Highlights the difference btw open and closed-loop strategies.

MFG with Major and Minor players – the problem

- A major player (states : X^0) and a continuum of identical minor players (states : X of measure μ), Here : *representative minor player*. Coupled dynamics :

$$dX_t^0 = b_0(t, X_t^0, \mu_t, \alpha_t^0)dt + \sigma_0(t, X_t^0, \mu_t, \alpha_t^0)dW_t^0$$
$$dX_t = b(t, X_t, \mu_t, X_t^0, \alpha_t, \alpha_t^0)dt + \sigma(t, X_t, \mu_t, X_t^0, \alpha_t, \alpha_t^0)dW_t$$

  where
  - W^0 and W : resp. m_0 and m-dim Brownian motions
  - b and σ deterministic functions
  - α^0 Major control, and α minor players strategies valued in A_0/A
  - Interaction :
    - 1st SDE depends on the distribution μ_t of the *solution* of the 2nd SDE
    - 2nd SDE depends on the *solution* and *control* of 1st SDE.

MFG Maj-Min : control problem

- The control problem of both players is to find the optimal paths $(\alpha_t^0)_t$, and $(\alpha_t)_t$, minimizing :

$$J^0(\alpha^0,\alpha) = \mathbb{E}\left[\int_0^T f^0(t, X_t^0, \mu_t, \alpha_t^0)dt + g^0(X_T^0, \mu_T)\right]$$

$$J(\alpha^0,\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, X_t^0, \alpha_t, \alpha_t^0)dt + g(X_T, \mu_T)\right]$$

- Issues :
  - Optimal control of Major player depends on the *distribution* of the state of the minor players.
  - Optimal control of Minor player depends on the state *and* control of the major player.
- Equilibrium : the distribution of the state of the representative minor player, conditional on the dynamics of the major player : $\mu_t = \mathcal{L}(X_t|W_{[0,t]}^0)$

Choice of strategy : open-loop vs. closed-loop

- The strategy of the players can be assumed *open-loop* :

$$\alpha_t^0 = \phi^0(t, W^0_{[0,t]}) \qquad \text{and} \qquad \alpha_t = \phi(t, W^0_{[0,t]}, W_{[0,t]})$$

  - Here, the control does not (directly) depend on the states (and control) of the other players.

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- Or assumed to be *closed-loop Markovian* :

$$\alpha_t^0 = \phi^0(t, X^0_t, \mu_t) \qquad \text{and} \qquad \alpha_t = \phi(t, X^0_t, \mu_t, X_t)$$

  - Idem, but the influence is only instantaneous.

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- Difference : matters a lot for the control of McKean Vlasov system

Best-response map and Nash equilibrium

- Nash equilibria : finding the fixed point of the 'best response map', here with two components :
  - Response of the major player $\phi^{0,*}$, taking as given all the minor players ϕ
  - Response of the deviating minor ϕ^{*}, taking as given the major player ϕ^{0} and the other minor players response ϕ.

Best-response map and Nash equilibrium

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  - Response of the major player $\phi^{0,*}$, taking as given all the minor players ϕ
  - Response of the deviating minor ϕ^*, taking as given the major player ϕ^0 and the other minor players response ϕ.
- Procedure : identify the best-response map before drawing the fixed point :
  - Major player :
$$\phi^{0,*}(\phi) = \underset{\alpha^0 \leftrightarrow \phi^0}{\text{arginf}}\, J^{\phi,0}(\alpha^0)$$
  - (Deviating) Minor player :
$$\phi^*(\phi^0, \phi) = \underset{\widetilde{\alpha} \leftrightarrow \tilde{\phi}}{\text{arginf}}\, J^{\phi^0,\phi}(\widetilde{\alpha})$$
  - Fixed point (Nash equilibrium) :
$$(\hat{\phi}^0, \hat{\phi}) = \left(\phi^{0,*}(\hat{\phi}), \phi^*(\hat{\phi}^0, \hat{\phi})\right)$$

Best-response map : a control problem for the *major* player

- Major player's best response requires to find the optimal control :

$$J^{\phi,0}(\alpha^0) = \mathbb{E}\left[\int_0^T f^0(t, X_t^0, \mu_t, \alpha_t^0)dt + g^0(X_T^0, \mu_T)\right]$$

  under the dynamics (open loop) :

$$dX_t^0 = b_0(t, X_t^0, \mu_t, \alpha_t^0)dt + \sigma_0(t, X_t^0, \mu_t, \alpha_t^0)dW_t^0$$

$$dX_t = b(t, X_t, \mu_t, X_t^0, \phi(t, W_{[0,t]}^0, W_{[0,t]}), \alpha_t^0)dt + \sigma(t, X_t, \mu_t, X_t^0, \phi(t, W_{[0,t]}^0, W_{[0,t]}), \alpha_t^0)dW_t$$

  where $\mu_t = \mathcal{L}(X_t | W_{[0,t]}^0)$

- Control of the McKean Vlasov type ! (conditional on the state of the representative minor players.
- Find the best response $\phi^{0,*}(\phi)$ (Stochastic Pontryagin Maximum Principle)

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under the dynamics (closed-loop) :

$$dX_t^0 = b_0(t, X_t^0, \mu_t, \alpha_t^0)dt + \sigma_0(t, X_t^0, \mu_t, \alpha_t^0)dW_t^0$$

$$dX_t = b(t, X_t, \mu_t, X_t^0, \phi(t, X_{[0,t]}, \mu_t, X_{[0,t]}^0), \alpha_t^0)dt + \sigma(t, X_t, \mu_t, X_t^0, \phi(t, X_{[0,t]}, \mu_t, X_{[0,t]}^0), \alpha_t^0)dW_t$$

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under the dynamics (Markovian feedback loop) :

$$dX_t^0 = b_0(t, X_t^0, \mu_t, \alpha_t^0)dt + \sigma_0(t, X_t^0, \mu_t, \alpha_t^0)dW_t^0$$
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- Control of the McKean Vlasov type ! (conditional on the state of the representative minor players.
- Find the best response $\phi^{0,*}(\phi)$ (Stochastic Pontryagin Maximum Principle)

Best-response map : a control problem for the *minor* player

- Deviating minor player's best response : optimal control
  - Suppose given the strategy of other (major and minor) players
  - i.e. the dynamics of X_t^0, X_t, through strategies ϕ^0 and ϕ
  - open loop, closed-loop or Markovian
- McKean Vlasov dynamics (open loop)

$$dX_t^0 = b_0\big(t, X_t^0, \mu_t, \phi^0(t, W^0_{[0,t]})\big)dt + \sigma_0\big(t, X_t^0, \mu_t, \phi^0(t, W^0_{[0,t]})\big)dW_t^0$$

$$dX_t = b\big(t, X_t, \mu_t, X_t^0, \phi(t, W^0_{[0,t]}, W_{[0,t]}), \phi^0(t, W^0_{[0,t]})\big)dt$$
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  - This yields $\mu_t = \mathcal{L}(X_t | W^0_{[0,t]})$
- Deviating minor player's best response : optimal control $\widetilde{\alpha}$:
- Standard control problem $\widetilde{\alpha} = \text{arginf}\, J^{\phi^0,\phi}$, but with random coefficients, i.e. conditional on these McKean-Vlasov dynamics.

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Best-response map : a control problem for the *minor* player

- Deviating minor player's best response : optimal control $\widetilde{\alpha}$:

$$J^{\phi^0,\phi}(\widetilde{\alpha}) = \mathbb{E}\left[\int_0^T f\big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\alpha}_t, \phi^0(t, W^0_{[0,t]})\big)dt + g(\widetilde{X}_T, \mu_T)\right]$$

  under its own dynamics (open loop) :

$$d\widetilde{X}_t = \big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\phi}(t, W^0_{[0,t]}, \widetilde{W}_{[0,t]}), \phi^0(t, W^0_{[0,t]})\big) \\ + \sigma\big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\phi}(t, W^0_{[0,t]}, \widetilde{W}_{[0,t]}), \phi^0(t, W^0_{[0,t]})\big)d\widetilde{W}_t$$

  where $\mu_t = \mathcal{L}(X_t | W^0_{[0,t]})$
- Standard control problem, but with random coefficients, i.e. conditional on the McKean-Vlasov dynamics μ_t & X_t^0 above.

Best-response map : a control problem for the *minor* player

- Deviating minor player's best response : optimal control $\widetilde{\alpha}$:

$$J^{\phi^0,\phi}(\widetilde{\alpha}) = \mathbb{E}\left[\int_0^T f\big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\alpha}_t, \phi^0(t, X^0_{[0,t]}, \mu_t)\big)dt + g(\widetilde{X}_T, \mu_T)\right]$$

under its own dynamics (closed loop) :

$$d\widetilde{X}_t = \big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\phi}(t, \widetilde{X}_{[0,t]}, \mu_t, X^0_{[0,t]}), \phi^0(t, X^0_{[0,t]}, \mu_t)\big) \\ + \sigma\big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\phi}(t, \widetilde{X}_{[0,t]}, \mu_t, X^0_{[0,t]}), \phi^0(t, X^0_{[0,t]}, \mu_t)\big)d\widetilde{W}_t$$

where $\mu_t = \mathcal{L}(X_t | W^0_{[0,t]})$

- Standard control problem, but with random coefficients, i.e. conditional on the McKean-Vlasov dynamics μ_t & X_t^0 above.

Best-response map : a control problem for the *minor* player

- Deviating minor player's best response : optimal control $\widetilde{\alpha}$:

$$J^{\phi^0,\phi}(\widetilde{\alpha}) = \mathbb{E}\left[\int_0^T f\big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\alpha}_t, \phi^0(t, X_t^0, \mu_t)\big)dt + g(\widetilde{X}_T, \mu_T)\right]$$

  under its own dynamics (Markovian) :

$$d\widetilde{X}_t = \big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\phi}(t, \widetilde{X}_t, \mu_t, X_t^0), \phi^0(t, X_t^0, \mu_t)\big) + \sigma\big(t, \widetilde{X}_t, \mu_t, X_t^0, \widetilde{\phi}(t, \widetilde{X}_t, \mu_t, X_t^0), \phi^0(t, X_t^0, \mu_t)\big)d\widetilde{W}_t$$

  where $\mu_t = \mathcal{L}(X_t | W^0_{[0,t]})$
- Standard control problem, but with random coefficients, i.e. conditional on the McKean-Vlasov dynamics μ_t & X_t^0 above.

Wrapping up

- Procedure : identify the best-response map before drawing the fixed point :
  - Major player :

$$\phi^{0,*}(\phi) = \underset{\alpha^0 \leftrightarrow \phi^0}{\operatorname{arginf}} J^{\phi,0}(\alpha^0)$$

  - (Deviating) Minor player :

$$\phi^{*}(\phi^0, \phi) = \underset{\widetilde{\alpha} \leftrightarrow \tilde{\phi}}{\operatorname{arginf}} J^{\phi^0,\phi}(\widetilde{\alpha})$$

  - Fixed point (Nash equilibrium) :

$$(\hat{\phi}^0, \hat{\phi}) = \left(\phi^{0,*}(\hat{\phi}), \phi^{*}(\hat{\phi}^0, \hat{\phi})\right)$$

LQ model

- LQ model :
  - Linear dynamics, depends linearly on the states (X_t^0 and/or X_t) and the first-moment of the distribution $\bar{X}_t = \mathbb{E}[X_t|\mathcal{F}_t^0]$
  - Quadratic costs in control α^0 or α and in the diff.$(X_t^0 - H_0\bar{X}_t)$ or $(X_t - HX_t^0 - H_1\bar{X}_t)$.
- Here we consider the mean-field limit ($N \to \infty$).

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- Here we consider the mean-field limit ($N \to \infty$).
- Issue and remedy :
  - Usually, resolution of control of McKean Vlasov dynamics : through *Stochastic Pontryagin Maximum Principle* (SPMP)
  - And involves the derivative of the Hamiltonian w.r.t. the measure μ
  - Here, the only info we have about the measure is the mean $\bar{X}_t$
  - *Remedy* : introduce $\bar{X}_t$ as a new variable
  - ... and solve the 'two variables' system.

LQ model : dynamics and strategy

- Linear dynamics :

$$dX_t^0 = \big(L_0 X_t^0 + B_0 \alpha_t^0 + F_0 \bar{X}_t\big) dt + D_0 dW_t^0$$
$$dX_t = \big(L X_t + B\alpha_t + F\bar{X}_t + G X_t^0\big) dt + D dW_t$$

  - Taking the expectation $\mathbb{E}[\cdot|\mathcal{F}_t^0]$ in the second equation, with $\bar{\alpha}_t$:

$$d\bar{X}_t = \big[(L+F)\bar{X}_t + B\bar{\alpha}_t + G X_t^0\big] dt$$

- We thus can rewrite the problem in terms of the couple $\mathbb{X}_t := (\bar{X}_t, X_t^0)$
  - Open-loop strategy : controls don't depend on a change in states
  - Closed-loop (Markovian) strategy : should restrict the reaction function

$$\alpha_t^0 = \phi^0(t, X_t^0, \bar{X}_t) = \phi_0^0(t) + \phi_1^0(t) X_t^0 + \phi_2^0(t) \bar{X}_t$$
$$\alpha_t = \phi(t, X_t, X_t^0, \bar{X}_t) = \phi_0(t) + \phi_1(t) X_t + \phi_2(t) X_t + \phi_3(t) \bar{X}_t$$

LQ model, 1st step : major player strategy

- Major player control problem :

$$\inf_{\alpha^0} \mathbb{E}\left[\int_0^T \mathbb{X}_t^T \mathbb{F}_0 \mathbb{X}_t + 2\mathbb{X}_t^T f_0 + \eta_0^T Q_0 \eta_0 + \alpha^{0\,T} R_0 \alpha^0 \, dt\right]$$

under the dynamics of $\mathbb{X}_t$:

$$d\mathbb{X}_t = (\mathbb{L}_0 \mathbb{X}_t + \mathbb{B}_0 \alpha_t^0 + \mathbb{B}\bar{\alpha}_t)dt + \mathbb{D}_0 dW_t^0$$

with the matrices :

$$\mathbb{X}_t = \begin{bmatrix} \bar{X}_t \\ X_t^0 \end{bmatrix}, \quad \mathbb{L}_0 = \begin{bmatrix} L+F & G \\ F_0 & L_0 \end{bmatrix}, \quad \mathbb{B}_0 = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\mathbb{F}_0 = \begin{bmatrix} H_0^T Q_0 H_0 & -H_0^T Q_0 \\ -Q_0 H_0 & Q_0 \end{bmatrix} \quad f_0 = \begin{bmatrix} H_0^T Q_0 \eta_0 \\ -Q_0 \eta_0 \end{bmatrix} \quad \mathbb{D}_0 = \begin{bmatrix} 0 \\ D_0 \end{bmatrix}$$

- The Hamiltonian (reduced) is given by :

$$H^{(r),\bar{\alpha}}(t,x,y,\alpha^0) = y^T(\mathbb{L}_0 x + \mathbb{B}_0 \alpha^0 + \mathbb{B}\bar{\alpha}_t) + x^T \mathbb{F}_0 x + 2x^T f_0 + \eta_0^T Q_0 \eta_0 + \alpha^{0\,T} R_0 \alpha^0$$

LQ model, 1st step : major player strategy (closed loop)

- Major player control problem (closed loop)

$$\inf_{\alpha^0} \mathbb{E}\left[\int_0^T \mathbb{X}_t^T \mathbb{F}_0 \mathbb{X}_t + 2\mathbb{X}_t^T f_0 + \eta_0^T Q_0 \eta_0 + \alpha^{0\,T} R_0 \alpha^0 \, dt\right]$$

under the dynamics of $\mathbb{X}_t$:

$$d\mathbb{X}_t = \left(\mathbb{L}_0^{(cl)} \mathbb{X}_t + \mathbb{B}_0 \alpha_t^0 + \mathbb{C}_0^{(cl)}(t)\right)dt + \mathbb{D}_0 dW_t^0$$

with the matrices :

$$\mathbb{X}_t = \begin{bmatrix} \bar{X}_t \\ X_t^0 \end{bmatrix} \quad \mathbb{L}_0^{(cl)} = \begin{bmatrix} L + B[\phi_1(t)+\phi_3(t)] + F & B\phi_2(t) + G \\ F_0 & L_0 \end{bmatrix} \quad \mathbb{B}_0 = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \quad \mathbb{D}_0 = \begin{bmatrix} 0 \\ D_0 \end{bmatrix}$$

$$\mathbb{C}_0^{(cl)} = \begin{bmatrix} B\phi_0(t) \\ 0 \end{bmatrix}, \qquad \mathbb{F}_0 = \begin{bmatrix} H_0^T Q_0 H_0 & -H_0^T Q_0 \\ -Q_0 H_0 & Q_0 \end{bmatrix} \quad f_0 = \begin{bmatrix} H_0^T Q_0 \eta_0 \\ -Q_0 \eta_0 \end{bmatrix}$$

- The Hamiltonian (reduced) is given by :

$$H^{(r),\phi}(t,x,y,\alpha^0) = y^T\left(\mathbb{L}_0^{(cl)} x + \mathbb{B}_0 \alpha^0 + \mathbb{C}_0^{(cl)}(t)\right) + x^T \mathbb{F}_0 x + 2x^T f_0 + \eta_0^T Q_0 \eta_0 + \alpha^{0\,T} R_0 \alpha^0$$

LQ model, 1st step : major player strategy

- Remarks (both open and closed loop) :
  - Adding a new variable $\bar{X}_t$ 'converts' the control of McKean-Vlasov dynamics into a 'standard' control problem.
  - However, it is still conditional on the move $\bar{\alpha}_t$ of the representative minor player
    - Conditionality is 'hidden' in the assumption $\alpha_t = \phi(\cdot)$

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    - Conditionality is 'hidden' in the assumption $\alpha_t = \phi(\cdot)$
- SPMP for McKean-Vlasov : Carmona, Delarue et al. (2015).
  - Provides a necessary and sufficient condition – if H is jointly convex in (x, α^0) :

$$\hat{\alpha}^0 \in \underset{\alpha^0}{\operatorname{arginf}}\, H^{(r),\bar{\alpha}}(\cdot, \alpha^0) \quad \text{(Isaacs cond°)} \qquad \Leftrightarrow \qquad \hat{\alpha}^0 = \alpha^{0,\star} \quad \text{(optimal control)}$$

  - Obtain a coupled FBSDE system (solution : exist & unique for LQ).
  - Adjoint BSDE :

$$dY_t = -(D_x H(t, X, \mu, Y, Z, \hat{\alpha}) + \widetilde{\mathbb{E}}\big[D_\mu H(t, \widetilde{X}, \mathcal{L}(X|\mathcal{F}^0), \widetilde{Y}, \hat{\alpha})(X)\big])dt + Z_t dW_t$$

LQ model, 1st step : open-loop equilibrium

- Isaacs condition :

$$\hat{\alpha}_t^0 = -\frac{1}{2} R_0^{-1} \mathbb{B}_0^T \mathbb{Y}_t$$

- Coupled FSBDE, conditional on minor agent action :

$$\begin{cases} d\mathbb{X}_t &= (\mathbb{L}_0 \mathbb{X}_t - \mathbb{B}_0 \frac{1}{2} R_0^{-1} \mathbb{B}_0^T \mathbb{Y}_t + \mathbb{B} \overline{\alpha}_t) dt + \mathbb{D}_0 dW_t^0 \\ d\mathbb{Y}_t &= -(\mathbb{L}_0^T \mathbb{Y}_t + 2\mathbb{F}_0 \mathbb{X}_t + 2f_0) dt + \mathbb{Z}_t dW_t^0 \\ \mathbb{Y}_T &= 0 \end{cases}$$

- Solving this FBSDE is equivalent to finding the best response of the major player

LQ model, 1st step : closed-loop equilibrium

- Isaacs condition :

$$\hat{\alpha}_t^0 = -\frac{1}{2} R_0^{-1} \mathbb{B}_0^T \mathbb{Y}_t$$

- Coupled FSBDE, conditional on minor agent reaction :

$$\begin{cases} d\mathbb{X}_t & = \left(\mathbb{L}_0^{(cl)} \mathbb{X}_t + \mathbb{B}_0 \frac{1}{2} R_0^{-1} \mathbb{B}_0^T \mathbb{Y}_t + \mathbb{C}_0^{(cl)}(t)\right) dt + \mathbb{D}_0 dW_t^0 \\ d\mathbb{Y}_t & = -(\mathbb{L}_0^T \mathbb{Y}_t + 2\mathbb{F}_0 \mathbb{X}_t + 2f_0) dt + \mathbb{Z}_t dW_t^0 \\ \mathbb{Y}_T & = 0 \end{cases}$$

- Solving this FBSDE is equivalent to finding the best response of the major player

LQ model, 1st step : closed-loop equilibrium

- Closed-loop specificity : we want to find a feedback form

$$\alpha_t^0 = \phi^0(\cdot) = \phi_0^0(t) + \phi_1^0(t)X_t^0 + \phi_2^0(t)\bar{X}_t$$

- Hypothesis on the decoupling field (affine) : $\mathbb{Y}_t = K_t\mathbb{X}_t + k_t$
- Finding the decoupling field :
  - Sketch for the method :
    - Use the ansatz $\mathbb{Y}_t = K_t\mathbb{X}_t + k_t$
    - Apply Ito's to get a second formula for dY
    - Identify the two Itô processes (the first being the BSDE above)
    - Obtain (two) Riccati's ODE for K_t and k_t
  - The optimal control is

$$\hat{\alpha}_t^0 = \underbrace{-\frac{1}{2}R_0^{-1}\mathbb{B}_0^T K_t}_{[\phi_2^0(t)\ \phi_1^0(t)]}\mathbb{X}_t \underbrace{-\frac{1}{2}R_0^{-1}\mathbb{B}_0^T k_t}_{\phi_0^0(t)}$$

LQ model, 2nd step : open-loop equilibrium

- The deviating minor player, consider a fixed strategy α^0 and α..

$$d\mathbb{X}_t = (\mathbb{L}_0\mathbb{X}_t + \mathbb{B}_0\alpha_t^0 + \mathbb{B}\bar{\alpha}_t)dt + \mathbb{D}_0 dW_t^0$$

- Given this state evolution (analogy : common noise !) :

LQ model, 2nd step : open-loop equilibrium

- The deviating minor player, consider a fixed strategy α^0 and α..

$$d\mathbb{X}_t = (\mathbb{L}_0\mathbb{X}_t + \mathbb{B}_0\alpha_t^0 + \mathbb{B}\bar{\alpha}_t)dt + \mathbb{D}_0 dW_t^0$$

- Given this state evolution (analogy : common noise !) :
- ... the deviating minor player solve the control problem :

$$\inf_{\widetilde{\alpha}} \mathbb{E}\left[\int_0^T (\widetilde{X}_t - [H_1, H]\mathbb{X}_t - \eta)^T Q(\widetilde{X}_t - [H_1, H]\mathbb{X}_t - \eta) + \widetilde{\alpha}_t^T R_0 \widetilde{\alpha}_t \, dt\right]$$

under its own dynamics :

$$d\widetilde{X}_t = \left(L\widetilde{X}_t + B\widetilde{\alpha}_t + [F, G]\mathbb{X}_t\right)dt + DdW_t$$

LQ model, 2nd step : open-loop equilibrium

- The deviating minor player, consider a fixed strategy α^0 and α..

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under its own dynamics :

$$d\widetilde{X}_t = \left(L\widetilde{X}_t + B\widetilde{\alpha}_t + [F, G]\mathbb{X}_t\right)dt + DdW_t$$

- Control problem : (Random) reduced Hamiltonian :

$$H^{(r),\alpha^0,\alpha}(t, \widetilde{x}, \widetilde{y}, \widetilde{\alpha}) = \widetilde{y}^T\left(L\widetilde{x} + B\widetilde{\alpha} + [F, G]\mathbb{X}_t\right) + (\widetilde{x} - [H_1, H]\mathbb{X}_t - \eta)^T Q(\widetilde{x} - [H_1, H]\mathbb{X}_t - \eta) + \widetilde{\alpha}^T R_0 \widetilde{\alpha}$$

LQ model, 2nd step : open-loop & SPMP

- Application of the SPMP to this 'standard' control problem :
  - Conditional on $\mathbb{X}_t$, $H^{(r),\alpha^0,\alpha}$ is almost-surely convex in $(\widetilde{x}, \widetilde{\alpha})$.
- Isaacs condition is N & S : $\widetilde{\alpha}_t^\star = -\frac{1}{2}R^{-1}B^T\widetilde{Y}_t$
- Coupled FBSDE, solved by $(\widetilde{X}_t, \widetilde{Y}_t)$:

$$\begin{cases} d\widetilde{X}_t & = \left(L\widetilde{X}_t - B\frac{1}{2}R^{-1}B^T\widetilde{Y}_t + [F,G]\widetilde{\mathbb{X}}_t\right)dt + DdW_t \\ d\widetilde{Y}_t & = -\left(L^T\widetilde{Y}_t + 2Q(X_t - [H_1, H]\widetilde{\mathbb{X}}_t - \eta)\right)dt + Z_t dW_t + Z_t^0 dW_t^0 \\ \widetilde{Y}_T & = 0 \end{cases}$$

using the *offline* dynamics :

$$d\widetilde{\mathbb{X}}_t = (\mathbb{L}_0\widetilde{\mathbb{X}}_t + \mathbb{B}_0\alpha_t^0 + \mathbb{B}\overline{\alpha}_t)dt + \mathbb{D}_0 dW_t^0$$

LQ model, 2nd step : closed-loop equilibrium

- The deviating minor player, consider a fixed strategy α^0 and α.

$$d\mathbb{X}_t = [\mathbb{L}^{(cl)}(t)\mathbb{X}_t + \mathbb{C}^{(cl)}(t)]dt + \mathbb{D}_0 dW_t^0$$

with matrices :

$$\mathbb{X}_t = \begin{bmatrix} \bar{X}_t \\ X_t^0 \end{bmatrix} \quad \mathbb{L}^{(cl)}(t) = \begin{bmatrix} L + B[\phi_1(t)+\phi_3(t)] + F & B\phi_2(t) + G \\ F_0 + B_0\phi_2^0 & L_0 + B_0\phi_1^0(t) \end{bmatrix} \quad \mathbb{C}^{(cl)}(t) = \begin{bmatrix} B\phi_0(t) \\ B\phi_0^0(t) \end{bmatrix}$$

- Given this state evolution (analogy : common noise !) :

LQ model, 2nd step : closed-loop equilibrium

- The deviating minor player, consider a fixed strategy α^0 and α.

$$d\mathbb{X}_t = [\mathbb{L}^{(cl)}(t)\mathbb{X}_t + \mathbb{C}^{(cl)}(t)]dt + \mathbb{D}_0 dW_t^0$$

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- Given this state evolution (analogy : common noise !) :
- ... the deviating minor player solve the control problem :

$$\inf_{\widetilde{\alpha}} \mathbb{E}\left[\int_0^T (\widetilde{X}_t - [H_1, H]\mathbb{X}_t - \eta)^T Q(\widetilde{X}_t - [H_1, H]\mathbb{X}_t - \eta) + \widetilde{\alpha}_t^T R_0 \widetilde{\alpha}_t \, dt\right]$$

under its own dynamics :

$$d\widetilde{X}_t = \left(L\widetilde{X}_t + B\widetilde{\alpha}_t + [F, G]\mathbb{X}_t\right)dt + DdW_t$$

LQ model, 2nd step : closed-loop and SPMP

- Control problem : (Random) reduced Hamiltonian :

$$H^{(r),\phi^0,\phi}(t,\widetilde{x},\widetilde{y},\widetilde{\alpha}) = \widetilde{y}^T\left(L\widetilde{X}_t + B\widetilde{\alpha}_t + [F,G]\mathbb{X}_t\right) + (\widetilde{x} - [H_1,H]\mathbb{X}_t - \eta)^T Q(\widetilde{x}_t - [H_1,H]\mathbb{X}_t - \eta) + \widetilde{\alpha}^T R_0 \widetilde{\alpha}$$

- Again, apply the SPMP to the control problem :
  - Isaacs condition is N & S : $\widetilde{\alpha}_t^\star = -\frac{1}{2}R^{-1}B^T\widetilde{Y}_t$
    - Conditional on $\mathbb{X}_t$, $H^{(r),\phi^0,\phi}$ is almost-surely convex in $(\widetilde{x},\widetilde{\alpha})$.

LQ model, 2nd step : closed-loop and SPMP

- Control problem : (Random) reduced Hamiltonian :

$$H^{(r),\phi^0,\phi}(t,\tilde{x},\tilde{y},\tilde{\alpha}) = \tilde{y}^T\left(L\tilde{X}_t + B\tilde{\alpha}_t + [F,G]\mathbb{X}_t\right) + (\tilde{x} - [H_1,H]\mathbb{X}_t - \eta)^T Q(\tilde{x}_t - [H_1,H]\mathbb{X}_t - \eta) + \tilde{\alpha}^T R_0 \tilde{\alpha}$$

- Again, apply the SPMP to the control problem :
  - Isaacs condition is N & S : $\tilde{\alpha}_t^\star = -\frac{1}{2}R^{-1}B^T\tilde{Y}_t$
    - Conditional on $\mathbb{X}_t$, $H^{(r),\phi^0,\phi}$ is almost-surely convex in $(\tilde{x},\tilde{\alpha})$.
- Coupled FBSDE, solved by $(\tilde{X}_t,\tilde{Y}_t)$:

$$\begin{cases} d\tilde{X}_t &= \left(L\tilde{X}_t - B\frac{1}{2}R^{-1}B^T\tilde{Y}_t + [F,G]\tilde{\mathbb{X}}_t\right)dt + DdW_t \\ d\tilde{Y}_t &= -\left(L^T\tilde{Y}_t + 2Q(X_t - [H_1,H]\tilde{\mathbb{X}}_t - \eta)\right)dt + Z_t dW_t + Z_t^0 dW_t^0 \\ \tilde{Y}_T &= 0 \\ d\mathbb{X}_t &= [\mathbb{L}^{(cl)}(t)\tilde{\mathbb{X}}_t + \mathbb{C}^{(cl)}(t)]dt + \mathbb{D}_0 dW_t^0 \end{cases}$$

  - Search for a feedback loop/decoupling field (ansatz & identificat°)
  - Obtain Riccati equations, etc.

LQ model, fixed point (open loop)

- Identify the fixed-point constraint, to insure Nash equilibrium :
  - Best-response of major player :

$$\alpha_t^0 = -\frac{1}{2} R_0^{-1} \mathbb{B}_0^T \mathbb{Y}_t \ \left(= \alpha^{0,*}(\alpha) \right)$$

    where $\mathbb{Y}_t$ solves the $(\mathbb{X}, \mathbb{Y})$ FBSDE (major player), and
  - Best response of the minor player α_t yields $\overline{\alpha}_t = \mathbb{E}[\alpha|\mathcal{F}^0]$ with :

$$\alpha_t = \widetilde{\alpha}_t = -\frac{1}{2} R^{-1} B^T \widetilde{Y}_t \ \left(= \widetilde{\alpha}^*(\alpha^0, \alpha) \right)$$

    where $\widetilde{Y}$ solves the $(\widetilde{X}_t, \widetilde{Y}_t)$-FBSDE (minor player), conditional on the state evolution $\widetilde{\mathbb{X}}_t$ (itself depending on α^0 and α).
- In equilibrium, $\mathbb{X}$ identifies with $\widetilde{\mathbb{X}}$

LQ model, fixed point (open loop)

- Identify the fixed-point constraint, to insure Nash equilibrium :
  - Best-response of major player :

$$\alpha_t^0 = -\frac{1}{2} R_0^{-1} \mathbb{B}_0^T \mathbb{Y}_t \; \left(= \alpha^{0,*}(\alpha) \right)$$

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    where $\widetilde{Y}$ solves the $(\widetilde{X}_t, \widetilde{Y}_t)$-FBSDE (minor player), conditional on the state evolution $\widetilde{\mathbb{X}}_t$ (itself depending on α^0 and α).
- In equilibrium, $\mathbb{X}$ identifies with $\widetilde{\mathbb{X}}$
- Theorem 1 : Verification theorem :
  - If the two systems of FBSDE admit a solution, then the LQ model has an open loop Nash equilibrium, and the strategies are given by these formulas.

LQ model, fixed point (closed-loop)

- Identify the fixed-point constraint, to insure Nash equilibrium :
  - Best-response of major player :

$$\begin{aligned}\alpha_t^{0,\star} &= \phi(t, X_t^0, \bar{X}_t) = \phi_0^0(t) + \phi_1^0(t)X_t^0 + \phi_2^0(t)\bar{X}_t \\ &= -\frac{1}{2}R_0^{-1}\mathbb{B}_0^T\mathbb{Y}_t \ \big(= \alpha^{0,*}(\alpha)\big) \\ &= -\frac{1}{2}R_0^{-1}\mathbb{B}_0^T K_t\mathbb{X}_t - \frac{1}{2}R_0^{-1}\mathbb{B}_0^T k_t\end{aligned}$$

  where $\mathbb{Y}_t$ solves the $(\mathbb{X}, \mathbb{Y})$ FBSDE (major player)
  - Best response of the minor player α_t yields $\overline{\alpha}_t = \mathbb{E}[\alpha|\mathcal{F}^0]$ with :

$$\begin{aligned}\widetilde{\alpha}_t^\star &= \phi(t, X_t, X_t^0, \bar{X}_t) = \phi_0(t) + \phi_1(t)X_t + \phi_2(t)X_t + \phi_3(t)\bar{X}_t \\ &= -\frac{1}{2}R^{-1}B^T\widetilde{Y}_t \ \big(= \widetilde{\alpha}^*(\alpha^0, \alpha)\big) \\ &= -\frac{1}{2}R^{-1}B^T[\mathbb{S}_t\mathbb{X}_t + S_tX_t + s_t]\end{aligned}$$

  where $\widetilde{Y}$ solves the $(\widetilde{X}_t, \widetilde{Y}_t)$-FBSDE (minor player), conditional on the state evolution $\widetilde{\mathbb{X}}_t$ (itself depending on α^0 and α).

LQ model, fixed point (closed-loop)

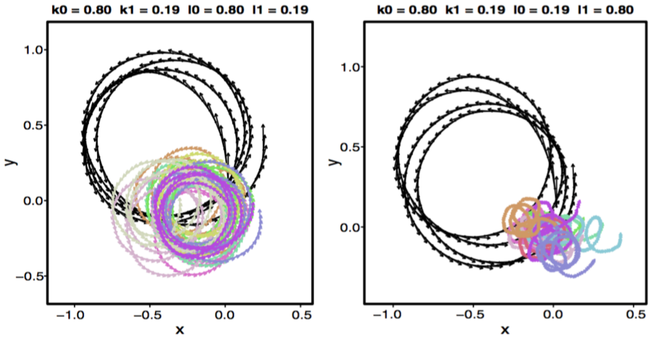
- Identify the fixed-point constraint, to insure Nash equilibrium.
- For major's player feedback : K_t and k_t solves specific Riccati's equation.
- For minor's player feedback : $\mathbb{S}_t$, S_t and s_t solves specific Riccati's equation.

$\Rightarrow$ More, the four Riccati ODE will be coupled :

- In equilibrium, $\mathbb{X}$ identifies with $\widetilde{\mathbb{X}}$
- Theorem 2 : Verification theorem :
  - If the system of Riccati's equation is well posed (and the two systems of FBSDE admit a solution), then the LQ model has an closed loop Nash equilibrium, and the strategies are given by the above formulas.

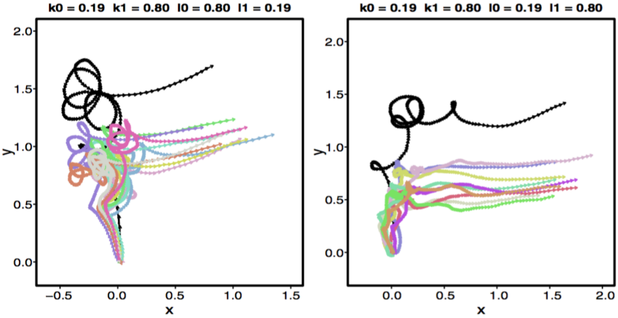
Application to Flocking models

- Flock composed by :
  - Leader (major player)
  - Followers (a mean-field of minor players)
  - Case 1 : leader does not consider so much the influence of the followers :



Application to Flocking models

- Flock composed by :
  - Leader (major player)
  - Followers (a mean-field of minor players)
  - Case 2 : leader care about the followers :



Discussion and conclusion

- Carmona and Wang (2016) : short and self-contained article
  - Statement of the problems, resolution of LQ case
  - Differences between Open and Closed loop Nash equilibria
- Carmona and Zhu (2016) is more complete, exhaustive article
- However, very interesting subject
  - Concentrate all the difficulties and challenges of Differential games, Mean Field Games and control of McKean Vlasov Dynamics.
- Article : pedagogical approach, a concrete example and link with many other theories.

- *Thank you for you attention !*

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